Exploring School Mathematics with Felix Klein

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Leitzel Lecture, 2010

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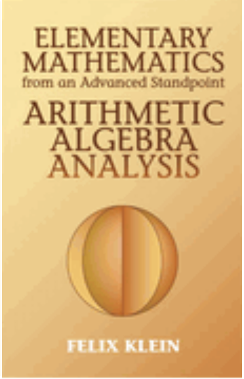
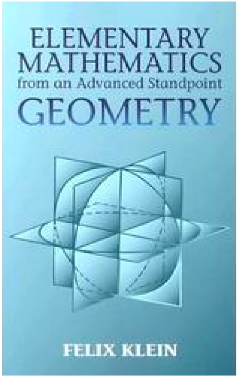




Sound familiar?

In recent years, a far reaching interest has arisen among university teachers of mathematics . . . directed toward a suitable training of candidates for the higher teaching positions. . . . For a long time . . . university [faculty] were concerned exclusively with their sciences, without giving a thought to the needs of the schools, without even caring to establish a connection with school mathematics.

The double discontinuity

The young university student [was] confronted with problems that did not suggest . . . the things with which he had been concerned at school. When, after finishing his course of study, he became a teacher . . . he was scarcely able to discern any connection between his task and his university mathematics . . .

The challenge

My task will always be to show you the mutual connection between problems in the various fields. In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: to draw from the great body of knowledge a living stimulus for your teaching.

A respect for teachers (if not for students)

What high regard one must have for the performance of elementary school teachers. Imagine what methodological training is necessary to indoctrinate over and over again a hundred thousand stupid, unprepared children with principles of arithmetic! Try it with your university training; you will not have great success!

A geometric view of solving equations

Quadratic equation:

$$t^2 + \lambda t + \mu = 0$$

Klein unwinds this into:

$$y + ux + v = 0$$

$$x = t, y = t^2; \qquad u = \lambda, v = \mu.$$

Two ways of seeing:

- In xy-space, a parameterized curve and a line.
- In uv-space, a parameterized family of lines and a point

The normal curve in uv-plane

Take one line from the family:

$$t^2 + ut + v = 0$$

And the infinitesimally neighboring line

$$(t + \Delta t)^2 + u(t + \Delta t) + v = 0$$

Subtract, divide by Δt and take limit (i.e., differentiate the family with respect to t):

$$2t + u = 0$$

Eliminate t:

$$v = \frac{u^2}{4}$$

Graph

Higher Degree Equations

Quartic equation

$$t^4 + \lambda t^2 + \mu t + \nu$$

Unwind this into

$$z + uy + vx + w = 0$$

$$x = t, y = t^2, z = t^4; \qquad u = \lambda, v = \mu, w = \nu$$

Graph

Device for Solving Higher Degree Equations

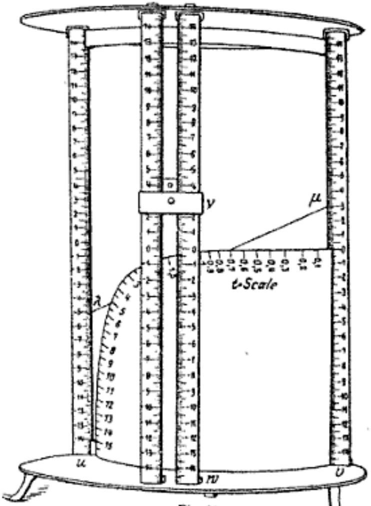


Fig. 31.

$$z + uy + vx + w = 0$$

$$z = \frac{x_3}{x_0}, y = \frac{x_2}{x_0}, x = \frac{x_1}{x_0}$$

$$x_3 + ux_2 + vx_1 + wx_0 = 0$$

The three verticals bars are the (projective) lines $(x_3, x_2, 0, 0)$, $(x_3, 0, x_1, 0)$, and $(x_3, 0, 0, x_0)$, marked with the coordinates $-\frac{x_3}{x_2}$, $-\frac{x_3}{x_1}$, and $-\frac{x_3}{x_0}$.

Dual point of view

System of planes in *uvw*-space

$$\begin{aligned} t^4 + t^2 \cdot u + t \cdot v + w &= 0 \\ 4t^3 + 2t \cdot u + v &= 0 \\ 12t^2 + 2 \cdot u &= 0 \end{aligned}$$

Curve

$$u = -6t^2, \quad v = 8t^3, \quad w = -3t^4$$

Intersection of

$$w + \frac{u^2}{12} = 0, \quad \frac{v^2}{8} + \frac{u^3}{27} = 0$$

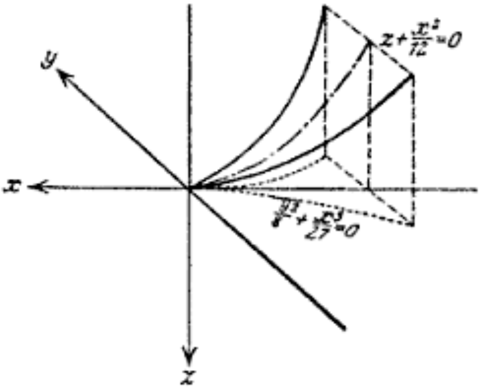
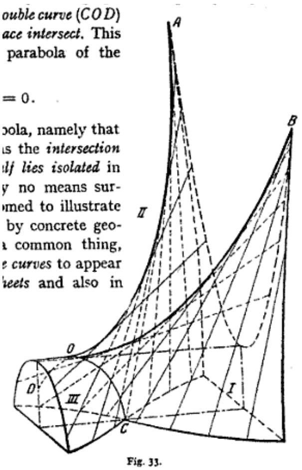


Fig. 32.

Well known to those who know it well



The envelope of these planes is the system of straight lines in which each plane $f(t) = 0$ meets the neighboring plane $f(t + \Delta t) = 0$. . . . But in order to obtain the normal curve we must seek the osculating configuration of the system of planes, i.e., the locus of the points of intersection of the three successive planes. This locus is, as you know, the cuspidal edge of that developable surface . . .

The double discontinuity revisited: Algebra

Where do students learn

- Why the product of two negative numbers is positive?
- Why you can "move the minus sign from top to bottom":

$$\frac{x+7}{-2} = \frac{-(x+7)}{2}.$$

- Why you can do this:

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right) = \frac{0.6}{3}(t_1+t_2+t_3).$$

The double discontinuity revisited: Analysis

Where do students connect the completeness axiom with their intuitive understanding of real numbers?

Education of teachers (or everybody but future Ph.D.s) (Wu, 1996)

- Only proofs of truly basic theorems are given, but whatever proofs are given should be complete and rigorous.
- In contrast with the normal courses which are relentlessly "forward-looking" (i.e., the far-better-things-to-come in graduate courses), considerable time should be devoted to "looking back".
- Keep the course on as concrete a level as possible, and introduce abstractions only when absolutely necessary.
- Ample historical background should be provided.
- Provide students with some perspective on each subject, including the presentation of surveys of advanced topics.
- Give motivation at every opportunity.

The Klein Project, 2010

- A project of ICMI celebrating 100 years of Klein's book.
- Focused on contemporary mathematics and contemporary applications.
- Provide a "living stimulus for teaching."
- http://kleinproject.org